INEQUALITIES

UNIT 0 INTRODUCTION

Inequalities are mathematical statements involving one or more inequality signs: >, <, $\geq$, $\leq$ or $\neq$. Readers are assumed to be familiar with the basic operations on inequalities: we may perform addition or subtraction on both sides of an inequality, but multiplication and division by a negative number will reverse the inequality sign. For instance, if $x \geq y$, then $-2x \leq -2y$.

The only axioms concerning inequalities are:

(1) Given a real number x, either $x > 0$, $x = 0$ or $x < 0$.

(2) If $a > 0$ and $b > 0$, then $a + b > 0$ and $ab > 0$.

(3) If $a > b$, then $a + c > b + c$.

All other inequalities can be derived from these axioms.

Of course, it will be tedious if we prove everything from the very beginning. Just like in deductive geometry, we will not prove every statement from the definitions and postulates; instead we rely on theorems and known results that have been built on these definitions and postulates and on other known theorems. Analogously, in proving an inequality, we seldom prove it by the above axioms directly; rather we will rely on some known facts. For instance, we want to prove that $a^2 \geq b(2a - b)$ for all real numbers a and b. Then we can apply the well-known fact $x^2 \geq 0$ for any real number x (which can be proved from the axioms — try it), so that $0 \leq (a-b)^2 = a^2 - b(2a-b)$, from which the result follows.

Therefore, we will study some well-known inequalities in Unit 1. Probably, the three most frequently used inequalities are the AM-GM inequality, the Cauchy-Schwarz inequality and the rearrangement inequality. But for completeness we shall include some more inequalities, most of which are generalizations and variations of the ‘Big Three’.

While the classical inequalities to be studied in Unit 1 are not difficult to understand, proving an inequality using the classical inequalities is often not easy. More precisely, an inequality sometimes cannot be proved by a direct application of one or more of the classical inequalities in

Unit 1. Instead, some techniques (e.g. substitutions, transformations, etc.) are often needed so that we can prove an inequality nicely (and not by brute force, which sometimes — but certainly not all the time — works). This shall be the focus of Unit 2.

Finally, inequalities is not a subject in its own. It is an algebraic tool which is used in many branches of mathematics. In Unit 3 we will study geometric inequalities, where we will study inequalities in geometric contexts. We shall again study some classical geometric inequalities, and then look at some problems to see the techniques involved.

While the subject of inequalities is put in an algebraic context, it should be noted that it is often concerned with optimization/extremum problems, where methods of calculus apply. Interested readers could refer to the notes ‘Differentiation (II)’ for the use of calculus in proving inequalities.